

Power series and integral forms of Lamé equation in the Weierstrass's form

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Abstract Lamé ordinary differential equation in the Weierstrass's form and Heun equation are of Fuchsian types with the three regular and one irregular singularities. Lamé equation in the Weierstrass's form is derived from Heun equation by changing all coefficients $\gamma = \delta = \varepsilon = \frac{1}{2}$, $a = p^{-2}$, $\beta = -(\alpha + 1)$, $q = -hp^{-2}$ and independent variable $x = sn^2(z, p)$. (see (1) in Ref.[16])

I consider the power series expansion of Lamé function in the Weierstrass's form and its integral forms applying three term recurrence formula[1]. I investigate asymptotic expansions of Lamé function for the cases of infinite series and polynomials. I will show how the power series expansion of Lamé functions in the Weierstrass's form can be converted to closed-form integrals for all cases of infinite series and polynomial. One interesting observation resulting from the calculations is the fact that a ${}_2F_1$ function recurs in each of sub-integral forms: the first sub-integral form contains zero term of $A'_n s$, the second one contains one term of A_n 's, the third one contains two terms of A_n 's, etc.

This paper is 7th out of 10 in series "Special functions and three term recurrence formula (3TRF)". See section 7 for all the papers in the series. Previous paper in series deals with the power series expansion and the integral formalism of Lamé equation in the algebraic form and its asymptotic behavior [19]. The next paper in the series describes the generating functions of Lamé equation in the Weierstrass's form[21].

Keywords Lamé equation · Integral form · Three term recurrence formula · Lamé polynomials · Ellipsoidal harmonic function

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1 Introduction

In 1837, Gabriel Lamé introduced second ordinary differential equation which has four regular singular points in the method of separation of variables applied to the Laplace equation in elliptic coordinates[10]. Various authors has called this equation as ‘Lamé equation’ or ‘ellipsoidal harmonic equation’[11].

Due to its mathematical complexity there is no analytic solution in closed forms of Lamé function[11, 12, 13]. Because its solution, in the algebraic form or in the Weierstrass’s form, was a form of a power series that is expressed as three term recurrence relation[12, 13]. In contrast, most of well-known special functions consist of two term recursion relation (Hypergeometric, Bessel, Legendre, Kummer functions, etc).

In my previous paper[2], applying three term recurrence formula.[1], I showed the power series expansion in closed forms of Lamé function in the algebraic form (infinite series and polynomial) including all higher terms of A_n ’s by applying three term recurrence formula.[1]. I obtained representations in form of contour integrals of Lamé function in the algebraic form and its asymptotic behavior of it and the boundary condition for x .

In this paper I will show the analytic solution of Lamé equation in the Weierstrass’s form. Its functions in the Weierstrass’s form appear as we apply the method of separation of variables to Laplace equation in an ellipsoidal coordinate system (Gabriel Lamé 1837[10]).

The Lamé equation in Weierstrass’s form in which is

$$\frac{\partial^2 y}{\partial z^2} = \{\alpha(\alpha + 1)\rho^2 sn^2(z, \rho) - h\}y(z) \quad (1)$$

where ρ , α and h are real parameters such that $0 < \rho < 1$ and $\alpha \geq -\frac{1}{2}$. If we take $sn^2(z, \rho) = \xi$ as independent variable, Lamé equation becomes

$$\frac{\partial^2 y}{\partial \xi^2} + \frac{1}{2} \left(\frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{\xi - \rho^{-2}} \right) \frac{\partial y}{\partial \xi} + \frac{-\alpha(\alpha + 1)\xi + h\rho^{-2}}{4\xi(\xi - 1)(\xi - \rho^{-2})} y(\xi) = 0 \quad (2)$$

This is an equation of Fuchsian type with the four regular singularities: $\xi = 0, 1, \rho^{-2}, \infty$. The first three, namely $0, 1, \rho^{-2}$, have the property that the corresponding exponents are $0, \frac{1}{2}$ which is the same as the case of Lamé equation in the algebraic form. In Ref.[2], Lamé equation of the algebraic form is

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{2} \left(\frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c} \right) \frac{\partial y}{\partial x} + \frac{-\alpha(\alpha + 1)x + q}{4(x - a)(x - b)(x - c)} y = 0 \quad (3)$$

If we compare (2) with (3), all coefficients on the above are exactly correspondent to the following way.

$$\begin{aligned} a &\longleftrightarrow 0 \\ b &\longleftrightarrow 1 \\ c &\longleftrightarrow \rho^{-2} \\ q &\longleftrightarrow h\rho^{-2} \\ x &\longleftrightarrow \xi = sn^2(z, \rho) \end{aligned} \quad (4)$$

We can another expression of Lamé function in the Weierstrass's form by using (4) in Ref.[2].

2 Power series

2.1 Polynomial in which makes B_n term terminated

The power series expansion of Lamé function in algebraic form for the polynomial in which makes B_n term terminated in Ref.[2] is

$$\begin{aligned}
 y(z) &= \sum_{n=0}^{\infty} y_n(z) \\
 &= c_0 z^{\lambda} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \right. \\
 &\quad + \sum_{i_0=0}^{\alpha_0} \left\{ \frac{(2a-b-c)(i_0 + \frac{\lambda}{2})^2 - a(\alpha_0 + \frac{\lambda}{2})(\alpha_0 + \frac{1}{4} + \frac{\lambda}{2}) + \frac{q}{24}}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \frac{1}{4} + \frac{\lambda}{2})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \right. \\
 &\quad \times \sum_{i_1=i_0}^{\alpha_1} \left\{ \frac{(-\alpha_1)_{i_1} (\alpha_1 + \frac{5}{4} + \lambda)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\frac{5}{4} + \frac{\lambda}{2})_{i_0}}{(-\alpha_1)_{i_0} (\alpha_1 + \frac{5}{4} + \lambda)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\frac{5}{4} + \frac{\lambda}{2})_{i_1}} \eta^{i_1} \right\} \Big\} \mu \\
 &\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{(2a-b-c)(i_0 + \frac{\lambda}{2})^2 - a(\alpha_0 + \frac{\lambda}{2})(\alpha_0 + \frac{1}{4} + \frac{\lambda}{2}) + \frac{q}{24}}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \frac{1}{4} + \frac{\lambda}{2})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \right. \\
 &\quad \times \prod_{k=1}^{n-1} \left\{ \sum_{i_k=i_{k-1}}^{\alpha_k} \frac{(2a-b-c)(i_k + \frac{k}{2} + \frac{\lambda}{2})^2 - a(\alpha_k + \frac{k}{2} + \frac{\lambda}{2})(\alpha_k + \frac{k}{2} + \frac{1}{4} + \frac{\lambda}{2}) + \frac{q}{24}}{(i_k + \frac{k}{2} + \frac{1}{2} + \frac{\lambda}{2})(i_k + \frac{k}{2} + \frac{1}{4} + \frac{\lambda}{2})} \right. \\
 &\quad \times \frac{(-\alpha_k)_{i_k} (\alpha_k + k + \frac{1}{4} + \lambda)_{i_k} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{3}{4} + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}}}{(-\alpha_k)_{i_{k-1}} (\alpha_k + k + \frac{1}{4} + \lambda)_{i_{k-1}} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{3}{4} + \frac{k}{2} + \frac{\lambda}{2})_{i_k}} \Big\} \\
 &\quad \times \sum_{i_n=i_{n-1}}^{\alpha_n} \frac{(-\alpha_n)_{i_n} (\alpha_n + n + \frac{1}{4} + \lambda)_{i_n} (1 + \frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}} (\frac{3}{4} + \frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}}}{(-\alpha_n)_{i_{n-1}} (\alpha_n + n + \frac{1}{4} + \lambda)_{i_{n-1}} (1 + \frac{n}{2} + \frac{\lambda}{2})_{i_n} (\frac{3}{4} + \frac{n}{2} + \frac{\lambda}{2})_{i_n}} \eta^{i_n} \Big\} \mu^n \Big\} \quad (5)
 \end{aligned}$$

where

$$\begin{cases} z = x - a \\ \eta = \frac{-(x-a)^2}{(a-b)(a-c)} \\ \mu = \frac{-(x-a)}{(a-b)(a-c)} \end{cases} \quad (6)$$

and

$$\begin{cases} \alpha = 2(2\alpha_i + i + \lambda) \text{ or } -2(2\alpha_i + i + \lambda) - 1 \text{ where } i, \alpha_i = 0, 1, 2, \dots \\ \alpha_i \leq \alpha_j \text{ only if } i \leq j \text{ where } i, j = 0, 1, 2, \dots \end{cases} \quad (7)$$

Put (4) in (5)-(7). And put $c_0 = 1$ as $\lambda=0$ and $c_0 = (-\rho^2)^{1/4}$ as $\lambda = \frac{1}{2}$ into them. Then, we obtain two independent solutions of Lamé equation in the Weierstrass's form.

$$\begin{aligned}
y(z) &= LF_{\alpha_j} \left(\alpha_j = \frac{1}{2} \left(\frac{\alpha}{2} - j \right) \text{ or } -\frac{1}{2} \left(\frac{\alpha}{2} + \frac{1}{2} + j \right) \Big|_{j=0,1,2,\dots}; \mu = -\rho^2 \operatorname{sn}^2(z, \rho); \eta = -\rho^2 \operatorname{sn}^4(z, \rho) \right) \\
&= \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \eta^{i_0} \\
&\quad + \mu \sum_{i_0=0}^{\alpha_0} \frac{-(1+\rho^{-2})i_0^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{4})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \sum_{i_1=i_0}^{\alpha_1} \frac{(-\alpha_1)_{i_1} (\alpha_1 + \frac{5}{4})_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{4})_{i_0}}{(-\alpha_1)_{i_0} (\alpha_1 + \frac{5}{4})_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{4})_{i_1}} \eta^{i_1} \\
&\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{-(1+\rho^{-2})i_0^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{4})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \right. \\
&\quad \times \prod_{k=1}^{n-1} \left(\sum_{i_k=i_{k-1}}^{\alpha_k} \frac{-(1+\rho^{-2})(i_k + \frac{k}{2})^2 + \frac{h}{24\rho^2}}{(i_k + \frac{k}{2} + \frac{1}{2})(i_k + \frac{k}{2} + \frac{1}{4})} \frac{(-\alpha_k)_{i_k} (\alpha_k + k + \frac{1}{4})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{3}{4} + \frac{k}{2})_{i_{k-1}}}{(-\alpha_k)_{i_{k-1}} (\alpha_k + k + \frac{1}{4})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{3}{4} + \frac{k}{2})_{i_k}} \right) \\
&\quad \times \left. \sum_{i_n=i_{n-1}}^{\alpha_n} \frac{(-\alpha_n)_{i_n} (\alpha_n + n + \frac{1}{4})_{i_n} (1 + \frac{n}{2})_{i_{n-1}} (\frac{3}{4} + \frac{n}{2})_{i_{n-1}}}{(-\alpha_n)_{i_{n-1}} (\alpha_n + n + \frac{1}{4})_{i_{n-1}} (1 + \frac{n}{2})_{i_n} (\frac{3}{4} + \frac{n}{2})_{i_n}} \eta^{i_n} \right\} \mu^n \quad (8)
\end{aligned}$$

$$\begin{aligned}
y(z) &= LS_{\alpha_j} \left(\alpha_j = \frac{1}{2} \left(\frac{\alpha}{2} - \frac{1}{2} - j \right) \text{ or } -\frac{1}{2} \left(\frac{\alpha}{2} + 1 + j \right) \Big|_{j=0,1,2,\dots}; \mu = -\rho^2 \operatorname{sn}^2(z, \rho); \eta = -\rho^2 \operatorname{sn}^4(z, \rho) \right) \\
&= \eta^{\frac{1}{4}} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{3}{4})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \eta^{i_0} \right. \\
&\quad + \mu \sum_{i_0=0}^{\alpha_0} \frac{-(1+\rho^{-2})(i_0 + \frac{1}{4})^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{3}{4})(i_0 + \frac{1}{2})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{3}{4})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \sum_{i_1=i_0}^{\alpha_1} \frac{(-\alpha_1)_{i_1} (\alpha_1 + \frac{7}{4})_{i_1} (\frac{7}{4})_{i_0} (\frac{3}{2})_{i_0}}{(-\alpha_1)_{i_0} (\alpha_1 + \frac{7}{4})_{i_0} (\frac{7}{4})_{i_1} (\frac{3}{2})_{i_1}} \eta^{i_1} \\
&\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{-(1+\rho^{-2})(i_0 + \frac{1}{4})^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{3}{4})(i_0 + \frac{1}{2})} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{3}{4})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \right. \\
&\quad \times \prod_{k=1}^{n-1} \left(\sum_{i_k=i_{k-1}}^{\alpha_k} \frac{-(1+\rho^{-2})(i_k + \frac{k}{2} + \frac{1}{4})^2 + \frac{h}{24\rho^2}}{(i_k + \frac{k}{2} + \frac{3}{4})(i_k + \frac{k}{2} + \frac{1}{2})} \frac{(-\alpha_k)_{i_k} (\alpha_k + k + \frac{3}{4})_{i_k} (\frac{5}{4} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_{k-1}}}{(-\alpha_k)_{i_{k-1}} (\alpha_k + k + \frac{3}{4})_{i_{k-1}} (\frac{5}{4} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_k}} \right) \\
&\quad \times \left. \sum_{i_n=i_{n-1}}^{\alpha_n} \frac{(-\alpha_n)_{i_n} (\alpha_n + n + \frac{3}{4})_{i_n} (\frac{5}{4} + \frac{n}{2})_{i_{n-1}} (1 + \frac{n}{2})_{i_{n-1}}}{(-\alpha_n)_{i_{n-1}} (\alpha_n + n + \frac{3}{4})_{i_{n-1}} (\frac{5}{4} + \frac{n}{2})_{i_n} (1 + \frac{n}{2})_{i_n}} \eta^{i_n} \right\} \mu^n \quad (9)
\end{aligned}$$

(8) is the first kind of independent solution of Lamé function for the polynomial in Weierstrass's form as $\alpha = 2(2\alpha_j + j)$ or $-2(2\alpha_j + j) - 1$ where $j, \alpha_j = 0, 1, 2, \dots$. And (9) is the second kind of independent solution of Lamé function in Weierstrass's form for the polynomial as $\alpha = 2(2\alpha_j + j) + 1$ or $-2(2\alpha_j + j + 1)$ where $j, \alpha_j = 0, 1, 2, \dots$.

2.2 Infinite series

The power series expansion of Lamé function in algebraic form for the infinite series in Ref.[2] is

$$\begin{aligned}
 y(z) &= \sum_{n=0}^{\infty} y_n(z) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\
 &= c_0 z^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{1}{4} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(2a-b-c)(i_0 + \frac{\lambda}{2})^2 - \frac{a}{24} \alpha(\alpha+1) + \frac{q}{24}}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \frac{1}{4} + \frac{\lambda}{2})} \right. \right. \\
 &\quad \times \frac{(-\frac{\alpha}{4} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{1}{4} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \sum_{i_1=i_0}^{\infty} \left\{ \frac{(-\frac{\alpha}{4} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{\alpha}{4} + \frac{3}{4} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\frac{5}{4} + \frac{\lambda}{2})_{i_0}}{(-\frac{\alpha}{4} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{3}{4} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\frac{5}{4} + \frac{\lambda}{2})_{i_1}} \eta^{i_1} \right\} \Bigg\} \mu \\
 &\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \frac{(2a-b-c)(i_0 + \frac{\lambda}{2})^2 - \frac{a}{24} \alpha(\alpha+1) + \frac{q}{24}}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \frac{1}{4} + \frac{\lambda}{2})} \frac{(-\frac{\alpha}{4} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{1}{4} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \right. \\
 &\quad \times \prod_{k=1}^{n-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(2a-b-c)(i_k + \frac{k}{2} + \frac{\lambda}{2})^2 - \frac{a}{24} \alpha(\alpha+1) + \frac{q}{24}}{(i_k + \frac{k}{2} + \frac{1}{2} + \frac{\lambda}{2})(i_k + \frac{k}{2} + \frac{1}{4} + \frac{\lambda}{2})} \right. \\
 &\quad \times \frac{(-\frac{\alpha}{4} + \frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{4} + \frac{\lambda}{2})_{i_k} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{k}{2} + \frac{3}{4} + \frac{\lambda}{2})_{i_{k-1}}}{(-\frac{\alpha}{4} + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{4} + \frac{\lambda}{2})_{i_{k-1}} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{k}{2} + \frac{3}{4} + \frac{\lambda}{2})_{i_k}} \Bigg\} \\
 &\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{n}{2} + \frac{\lambda}{2})_{i_n} (\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{4} + \frac{\lambda}{2})_{i_n} (1 + \frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}} (\frac{n}{2} + \frac{3}{4} + \frac{\lambda}{2})_{i_{n-1}}}{(-\frac{\alpha}{4} + \frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}} (\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{4} + \frac{\lambda}{2})_{i_{n-1}} (1 + \frac{n}{2} + \frac{\lambda}{2})_{i_n} (\frac{n}{2} + \frac{3}{4} + \frac{\lambda}{2})_{i_n}} \eta^{i_n} \Bigg\} \mu^n \Bigg\} \quad (10)
 \end{aligned}$$

Put (4) in (10). And put $c_0 = 1$ as $\lambda=0$ and $c_0 = (-\rho^2)^{1/4}$ as $\lambda = \frac{1}{2}$ into them. Then, we obtain two independent solutions of Lamé equation in the Weierstrass's form.

$$\begin{aligned}
 y(z) &= LF(\mu = -\rho^2 sn^2(z, \rho); \eta = -\rho^2 sn^4(z, \rho)) \\
 &= \sum_{i_0=0}^{\infty} \frac{(-\frac{\alpha}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \eta^{i_0} \\
 &\quad + \mu \sum_{i_0=0}^{\infty} \frac{-(1+\rho^{-2})i_0^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{4})} \frac{(-\frac{\alpha}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{1}{2})_{i_1} (\frac{\alpha}{4} + \frac{3}{4})_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{4})_{i_0}}{(-\frac{\alpha}{4} + \frac{1}{2})_{i_0} (\frac{\alpha}{4} + \frac{3}{4})_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{4})_{i_1}} \eta^{i_1} \\
 &\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \frac{-(1+\rho^{-2})i_0^2 + \frac{h}{24\rho^2}}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{4})} \frac{(-\frac{\alpha}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{4})_{i_0}}{(\frac{3}{4})_{i_0} (1)_{i_0}} \right. \\
 &\quad \times \prod_{k=1}^{n-1} \left(\sum_{i_k=i_{k-1}}^{\infty} \frac{-(1+\rho^{-2})(i_k + \frac{k}{2})^2 + \frac{h}{24\rho^2}}{(i_k + \frac{k}{2} + \frac{1}{2})(i_k + \frac{k}{2} + \frac{1}{4})} \frac{(-\frac{\alpha}{4} + \frac{k}{2})_{i_k} (\frac{\alpha}{4} + \frac{1}{4} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{3}{4} + \frac{k}{2})_{i_{k-1}}}{(-\frac{\alpha}{4} + \frac{k}{2})_{i_{k-1}} (\frac{\alpha}{4} + \frac{1}{4} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{3}{4} + \frac{k}{2})_{i_k}} \right) \\
 &\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{n}{2})_{i_n} (\frac{\alpha}{4} + \frac{1}{4} + \frac{n}{2})_{i_n} (1 + \frac{n}{2})_{i_{n-1}} (\frac{3}{4} + \frac{n}{2})_{i_{n-1}}}{(-\frac{\alpha}{4} + \frac{n}{2})_{i_{n-1}} (\frac{\alpha}{4} + \frac{1}{4} + \frac{n}{2})_{i_{n-1}} (1 + \frac{n}{2})_{i_n} (\frac{3}{4} + \frac{n}{2})_{i_n}} \eta^{i_n} \Bigg\} \mu^n \quad (11)
 \end{aligned}$$

$$\begin{aligned}
y(z) &= LS(\mu = -\rho^2 sn^2(z, \rho); \eta = -\rho^2 sn^4(z, \rho)) \\
&= \eta^{\frac{1}{4}} \left\{ \sum_{i_0=0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{1}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{2})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \eta^{i_0} \right. \\
&\quad + \mu \sum_{i_0=0}^{\infty} \frac{-(1+\rho^{-2})(i_0 + \frac{1}{4})^2 + \frac{h}{2^4 \rho^2}}{(i_0 + \frac{3}{4})(i_0 + \frac{1}{2})} \frac{(-\frac{\alpha}{4} + \frac{1}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{2})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{3}{4})_{i_1} (\frac{\alpha}{4} + 1)_{i_1} (\frac{7}{4})_{i_0} (\frac{3}{2})_{i_0}}{(-\frac{\alpha}{4} + \frac{3}{4})_{i_0} (\frac{\alpha}{4} + 1)_{i_0} (\frac{7}{4})_{i_1} (\frac{3}{2})_{i_1}} \eta^{i_1} \\
&\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \frac{-(1+\rho^{-2})(i_0 + \frac{1}{4})^2 + \frac{h}{2^4 \rho^2}}{(i_0 + \frac{3}{4})(i_0 + \frac{1}{2})} \frac{(-\frac{\alpha}{4} + \frac{1}{4})_{i_0} (\frac{\alpha}{4} + \frac{1}{2})_{i_0}}{(\frac{5}{4})_{i_0} (1)_{i_0}} \right. \\
&\quad \times \prod_{k=1}^{n-1} \left(\sum_{i_k=i_{k-1}}^{\infty} \frac{-(1+\rho^{-2})(i_k + \frac{k}{2} + \frac{1}{4})^2 + \frac{h}{2^4 \rho^2}}{(i_k + \frac{k}{2} + \frac{3}{4})(i_k + \frac{k}{2} + \frac{1}{2})} \right. \\
&\quad \times \frac{(-\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{4})_{i_k} (\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{2})_{i_k} (\frac{5}{4} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_{k-1}}}{(-\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{4})_{i_{k-1}} (\frac{\alpha}{4} + \frac{k}{2} + \frac{1}{2})_{i_{k-1}} (\frac{5}{4} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_k}} \Big) \\
&\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{4})_{i_n} (\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{2})_{i_n} (\frac{5}{4} + \frac{n}{2})_{i_{n-1}} (1 + \frac{n}{2})_{i_{n-1}}}{(-\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{4})_{i_{n-1}} (\frac{\alpha}{4} + \frac{n}{2} + \frac{1}{2})_{i_{n-1}} (\frac{5}{4} + \frac{n}{2})_{i_n} (1 + \frac{n}{2})_{i_n}} \eta^{i_n} \Big\} \mu^n \Big\} \quad (12)
\end{aligned}$$

(11) is the first kind of independent solution of Lamé function in Weierstrass's form for the infinite series. And (12) is the second kind of independent solution of Lamé function in Weierstrass's form for the infinite series.

3 Integral Formalism

3.1 Polynomial in which makes B_n term terminated

The representation in the form of integral of Lamé function in algebraic form for the polynomial in which makes B_n term terminated in Ref.[2] is

$$\begin{aligned}
y(z) &= \sum_{n=0}^{\infty} y_n(z) \\
&= c_0 z^{\lambda} \left\{ \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \right. \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-\frac{5}{2}+\lambda)} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-2+\lambda)} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftrightarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k}))^{-(n-k+\frac{1}{4}+\lambda)} \\
&\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftrightarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k})} \right)^{\alpha_{n-k}} \\
&\quad \times \left\{ (2a - b - c) \overleftrightarrow{W}_{n-k,n}^{-\frac{1}{2}(n-k-1+\lambda)} \left(\overleftrightarrow{W}_{n-k,n} \partial_{\overleftrightarrow{W}_{n-k,n}} \right)^2 \overleftrightarrow{W}_{n-k,n}^{\frac{1}{2}(n-k-1+\lambda)} \right. \\
&\quad \left. \left. - a \left(\alpha_{n-k-1} + \frac{1}{2}(n-k-1+\lambda) \right) \left(\alpha_{n-k-1} + \frac{1}{2}(n-k-\frac{1}{2}+\lambda) \right) + \frac{q}{2^4} \right\} \right\} \\
&\quad \left. \times \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \overleftrightarrow{W}_{1,n}^{i_0} \right\} \mu^n \Bigg\} \quad (13)
\end{aligned}$$

where

$$\overleftrightarrow{W}_{i,j} = \begin{cases} \frac{1}{(v_i-1)} \frac{\overleftrightarrow{W}_{i+1,j} v_i t_i u_i}{1 - \overleftrightarrow{W}_{i+1,j} v_i (1-t_i)(1-u_i)} \\ \eta \quad \text{only if } i > j \end{cases} \quad (14)$$

Put (4) in (13).

$$\begin{aligned}
y(z) &= \sum_{n=0}^{\infty} y_n(z) \\
&= c_0 z^\lambda \left\{ \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \right. \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-\frac{5}{2}+\lambda)} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-2+\lambda)} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftrightarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k}) (1 - u_{n-k}))^{-(n-k+\frac{1}{4}+\lambda)} \\
&\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftrightarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k}) (1 - u_{n-k})} \right)^{\alpha_{n-k}} \\
&\quad \times \left\{ -(1 + \rho^{-2}) \overleftrightarrow{w}_{n-k,n}^{-\frac{1}{2}(n-k-1+\lambda)} \left(\overleftrightarrow{w}_{n-k,n} \partial_{\overleftrightarrow{w}_{n-k,n}} \right)^2 \overleftrightarrow{w}_{n-k,n}^{\frac{1}{2}(n-k-1+\lambda)} + \frac{h}{2^4 \rho^2} \right\} \\
&\quad \times \sum_{i_0=0}^{\alpha_0} \frac{(-\alpha_0)_{i_0} (\alpha_0 + \frac{1}{4} + \lambda)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \overleftrightarrow{w}_{1,n}^{i_0} \left. \right\} \mu^n \Bigg\} \quad (15)
\end{aligned}$$

Put $c_0 = 1$ as $\lambda = 0$ and $c_0 = (-\rho^2)^{1/4}$ as $\lambda = \frac{1}{2}$ in (15).

$$\begin{aligned}
y(z) &= LF_{\alpha_j} \left(\alpha_j = \frac{1}{2} \left(\frac{\alpha}{2} - j \right) \text{ or } -\frac{1}{2} \left(\frac{\alpha}{2} + \frac{1}{2} + j \right) \Big|_{j=0,1,2,\dots}; \mu = -\rho^2 sn^2(z, \rho); \eta = -\rho^2 sn^4(z, \rho) \right) \\
&= {}_2F_1 \left(-\alpha_0, \alpha_0 + \frac{1}{4}; \frac{3}{4}; \eta \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-\frac{5}{2})} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-2)} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftrightarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k}) (1 - u_{n-k}))^{-(n-k+\frac{1}{4})} \\
&\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftrightarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k}) (1 - u_{n-k})} \right)^{\alpha_{n-k}} \\
&\quad \times \left\{ -(1 + \rho^{-2}) \overleftrightarrow{w}_{n-k,n}^{-\frac{1}{2}(n-k-1)} \left(\overleftrightarrow{w}_{n-k,n} \partial_{\overleftrightarrow{w}_{n-k,n}} \right)^2 \overleftrightarrow{w}_{n-k,n}^{\frac{1}{2}(n-k-1)} + \frac{h}{2^4 \rho^2} \right\} \\
&\quad \times {}_2F_1 \left(-\alpha_0, \alpha_0 + \frac{1}{4}; \frac{3}{4}; \overleftrightarrow{w}_{1,n} \right) \Bigg\} \mu^n \quad (16)
\end{aligned}$$

(16) is the integral formalism of the first kind of independent solution of Lamé function in Weierstrass's form for the polynomial as $\alpha = 2(2\alpha_j + j)$ or $-2(2\alpha_j + j) - 1$

where $j, \alpha_j = 0, 1, 2, \dots$.

$$\begin{aligned}
 y(z) &= LS_{\alpha_j} \left(\alpha_j = \frac{1}{2} \left(\frac{\alpha}{2} - \frac{1}{2} - j \right) \text{ or } -\frac{1}{2} \left(\frac{\alpha}{2} + 1 + j \right) \Big|_{j=0,1,2,\dots}; \mu = -\rho^2 \operatorname{sn}^2(z, \rho); \eta = -\rho^2 \operatorname{sn}^4(z, \rho) \right) \\
 &= \eta^{\frac{1}{4}} \left\{ {}_2F_1 \left(-\alpha_0, \alpha_0 + \frac{3}{4}; \frac{5}{4}; \eta \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-2)} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-\frac{3}{2})} \right. \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k}))^{-(n-k+\frac{3}{4})} \\
 &\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k})} \right)^{\alpha_{n-k}} \\
 &\quad \times \left\{ -(1 + \rho^{-2}) \overleftarrow{w}_{n-k,n}^{-\frac{1}{2}(n-k-\frac{1}{2})} \left(\overleftarrow{w}_{n-k,n} \partial_{\overleftarrow{w}_{n-k,n}} \right)^2 \overleftarrow{w}_{n-k,n}^{\frac{1}{2}(n-k-\frac{1}{2})} + \frac{h}{24\rho^2} \right\} \\
 &\quad \left. \left. \times {}_2F_1 \left(-\alpha_0, \alpha_0 + \frac{3}{4}; \frac{5}{4}; \overleftarrow{w}_{1,n} \right) \right\} \mu^n \right\} \quad (17)
 \end{aligned}$$

(17) is the integral formalism of the second kind of independent solution of Lamé function in Weierstrass's form for the polynomial as $\alpha = 2(2\alpha_j + j) + 1$ or $-2(2\alpha_j + j + 1)$ where $j, \alpha_j = 0, 1, 2, \dots$.

3.2 Infinite series

The representation in the form of integral of Lamé function in algebraic form for the infinite series in Ref.[2] is

$$\begin{aligned}
 y(z) &= \sum_{n=0}^{\infty} y_n(z) \\
 &= c_0 z^{\lambda} \left\{ \sum_{i_0=0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{1}{4} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-\frac{5}{2}+\lambda)} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-2+\lambda)} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k}))^{-(n-k+\frac{1}{4}+\lambda)} \\
 &\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftarrow{w}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k})} \right)^{\frac{1}{2}(\frac{\alpha}{2}-n+k-\lambda)} \\
 &\quad \times \left\{ (2a-b-c) \overleftarrow{w}_{n-k,n}^{-\frac{1}{2}(n-k-1+\lambda)} \left(\overleftarrow{w}_{n-k,n} \partial_{\overleftarrow{w}_{n-k,n}} \right)^2 \overleftarrow{w}_{n-k,n}^{\frac{1}{2}(n-k-1+\lambda)} \right. \\
 &\quad \left. \left. - \frac{a}{24} \alpha(\alpha+1) + \frac{q}{24} \right\} \right\} \sum_{i_0=0}^{\infty} \frac{(-\frac{\alpha}{4} + \frac{\lambda}{2})_{i_0} (\frac{\alpha}{4} + \frac{1}{4} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\frac{3}{4} + \frac{\lambda}{2})_{i_0}} \overleftarrow{w}_{1,n}^{i_0} \left. \right\} \mu^n \quad (18)
 \end{aligned}$$

Put (4) in (18). And put $c_0 = 1$ as $\lambda = 0$ and $c_0 = (-\rho^2)^{1/4}$ as $\lambda = \frac{1}{2}$ into them. Then, we obtain two independent solutions of Lamé equation in the Weierstrass's form.

$$\begin{aligned}
y(x) &= LF(\mu = -\rho^2 sn^2(z, \rho); \eta = -\rho^2 sn^4(z, \rho)) \\
&= {}_2F_1\left(-\frac{\alpha}{4}, \frac{\alpha}{4} + \frac{1}{4}; \frac{3}{4}; \eta\right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-\frac{5}{2})} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-2)} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k}))^{-(n-k+\frac{1}{4})} \\
&\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k})} \right)^{\frac{1}{2}(\frac{\alpha}{2}-n+k)} \\
&\quad \times \left. \left\{ -(1 + \rho^{-2}) \overleftarrow{W}_{n-k,n}^{-\frac{1}{2}(n-k-1)} \left(\overleftarrow{W}_{n-k,n} \partial_{\overleftarrow{W}_{n-k,n}} \right)^2 \overleftarrow{W}_{n-k,n}^{\frac{1}{2}(n-k-1)} + \frac{h}{2^4 \rho^2} \right\} \right\} \\
&\quad \times {}_2F_1\left(-\frac{\alpha}{4}, \frac{\alpha}{4} + \frac{1}{4}; \frac{3}{4}; \overleftarrow{W}_{1,n}\right) \Big\} \mu^n \tag{19}
\end{aligned}$$

$$\begin{aligned}
y(x) &= LS(\mu = -\rho^2 sn^2(z, \rho); \eta = -\rho^2 sn^4(z, \rho)) \\
&= \eta^{\frac{1}{4}} \left\{ {}_2F_1\left(-\frac{\alpha}{4} + \frac{1}{4}, \frac{\alpha}{4} + \frac{1}{2}; \frac{5}{4}; \eta\right) \right. \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left(\int_0^1 dt_{n-k} t_{n-k}^{\frac{1}{2}(n-k-2)} \int_0^1 du_{n-k} u_{n-k}^{\frac{1}{2}(n-k-\frac{3}{2})} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-k} \frac{1}{v_{n-k}} (1 - \overleftarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k}))^{-(n-k+\frac{3}{4})} \\
&\quad \times \left(\frac{(v_{n-k}-1)}{v_{n-k}} \frac{1}{1 - \overleftarrow{W}_{n-k+1,n} v_{n-k} (1 - t_{n-k})(1 - u_{n-k})} \right)^{\frac{1}{2}(\frac{\alpha}{2}-n+k-\frac{1}{2})} \\
&\quad \times \left. \left\{ -(1 + \rho^{-2}) \overleftarrow{W}_{n-k,n}^{-\frac{1}{2}(n-k-\frac{1}{2})} \left(\overleftarrow{W}_{n-k,n} \partial_{\overleftarrow{W}_{n-k,n}} \right)^2 \overleftarrow{W}_{n-k,n}^{\frac{1}{2}(n-k-\frac{1}{2})} + \frac{h}{2^4 \rho^2} \right\} \right\} \\
&\quad \times {}_2F_1\left(-\frac{\alpha}{4} + \frac{1}{4}, \frac{\alpha}{4} + \frac{1}{2}; \frac{5}{4}; \overleftarrow{W}_{1,n}\right) \Big\} \mu^n \tag{20}
\end{aligned}$$

(19) is the integral formalism of the first kind of independent solution of Lamé function in Weierstrass's form for the infinite series. And (20) is the integral formalism of the second kind of independent solution of Lamé function in Weierstrass's form for the infinite series.

4 Asymptotic behavior of the function $y(\xi = sn^2(z, \rho))$ and the boundary condition for $sn^2(z, \rho)$

4.1 Infinite series

The condition of convergence in Lamé function in the algebraic form for the infinite series and its asymptotic function in Ref.[2] are

$$\lim_{n \gg 1} y(z) = \frac{1}{1 + \left(\frac{(x-a)^2}{(a-b)(a-c)} + \frac{(2a-b-c)(x-a)}{(a-b)(a-c)} \right)} \quad (21)$$

$$\left| \frac{(x-a)^2}{(a-b)(a-c)} + \frac{(2a-b-c)(x-a)}{(a-b)(a-c)} \right| < 1 \quad (22)$$

Put (4) in (21) and (22). And its asymptotic function and the boundary condition of $\xi = sn^2(z, \rho)$ for the infinite series of Lamé function is

$$\lim_{n \gg 1} y(sn^2(z, \rho)) = \frac{1}{1 + \rho^2 sn^4(z, \rho) - (1 + \rho^2) sn^2(z, \rho)} \quad (23)$$

$$\text{where } |(1 + \rho^2) sn^2(z, \rho) - \rho^2 sn^4(z, \rho)| < 1$$

More precisely, as $0 < \rho^{-2} < \left(\frac{1 + \rho^{-2}}{2} \right)^2$

$$0 < sn^2(z, \rho) < \rho^{-2} \quad (24)$$

4.2 The case of $\rho \approx 0$

Let assume that ρ is approximately close to 0. But $\rho \neq 0$. Then B_n terms are negligible.

The condition of convergence in Lamé function in the algebraic form for the case of $2a - b - c \gg 1$ or $2a - b - c \ll -1$ and its asymptotic function in Ref.[2] are

$$\lim_{n \gg 1} y(z) = \frac{1}{1 + \left(\frac{(2a-b-c)(x-a)}{(a-b)(a-c)} \right)} \quad \text{where } a \neq b \text{ and } a \neq c \quad (25)$$

$$\left| \frac{(2a-b-c)(x-a)}{(a-b)(a-c)} \right| < 1 \quad (26)$$

Put (4) in (25) and (26). And its asymptotic function and the boundary condition of $\xi = sn^2(z, \rho)$ for the polynomial in which makes B_n term terminated is

$$\lim_{n \gg 1} y(sn^2(z, \rho)) = \frac{1}{1 - (1 + \rho^2) sn^2(z, \rho)} \quad (27)$$

The condition of convergence of $sn^2(z, \rho)$ is

$$0 < sn^2(z, \rho) < \frac{1}{1 + \rho^2} \quad (28)$$

5 Application

Lame equation appears elsewhere in mathematical physics. For example, Recently, in “Droplet nucleation and domain wall motion in a bounded interval”[4], the authors investigate an extended model (a classical GinzburgLandau model) of noise-induced magnetization reversal. Lame equation arises in some specific boundary conditions.(see (8), (9) in Ref.[4]. In (9) its solution consists of the Jacobi eta, theta, and zeta functions according to Hermite’s solution of the Lame equation.) In “Group Theoretical Properties and Band Structure of the Lame Hamiltonian”[9], the authors represent a group theoretical analysis of the Lame equation, which is an example of a SGA band structure problem for $su(2)$ and $su(1, 1)$. (see (1), (10), (13), (14), (28), (29), (33), (38) in Ref.[9]) Applying three term recurrence formula[1], we can obtain the power series expansion in closed forms and asymptotic behaviors of Lame function analytically. And it might be possible to obtain specific eigenvalues for the Lame Hamiltonian. Again Lame equation is applicable to diverse areas such as theory of the stability analysis of static configurations in Josephson junctions [5], the computation of the distance-redshift relation in inhomogeneous cosmologies[6], magnetostatic problems in triaxial ellipsoids[7] and etc.

6 Conclusion

From the above all, applying three term recurrence formula [1], I show the power series expansion in closed forms of Lame function in the Weierstrass’s form (infinite series and polynomials) and its integral forms. I show that a ${}_2F_1$ function recurs in each of sub-integral forms of Lame function in the Weierstrass’s form: the first sub-integral form contains zero term of A_n ’s, the second one contains one term of A_n ’s, the third one contains two terms of A_n ’s, etc. And I show asymptotic expansions of Lame function for infinite series and the special case as $\rho \approx 0$. Since we obtain the closed integral forms of Lame function in the Weierstrass’s form, Lame function is able to be transformed to other well-known special functions analytically; hypergeometric function, Mathieu function, Lame function, confluent forms of Heun function and etc.

Various authors argue that the value of $h\rho^{-2}$ can be chosen properly such that the Lame function is not an infinite series but a polynomial as α parameters of Lame functions is a positive integer. In the analysis of the three term recurrence formula [1], since α is $2(2\alpha_i + i + \lambda)$ or $-2(2\alpha_i + i + \lambda) - 1$ as $i = 0, 1, 2, \dots$, Lame functions will be polynomial in which makes B_n term terminated from (8) and (9). λ is the indicial roots which are 0 or $\frac{1}{2}$, then all possible α is $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$.

According to Erdelyi (1940[8]), “there is no corresponding representation of simple integral formalisms of the solutions in ordinary linear differential equations with four regular singularities; Heun equation, Lame equation and Mathieu equation. It appears that the theory of integral equations connected with periodic solutions of Lame equation is not as complete as the corresponding theory of integral representations of, say, Legendre functions.” The reason, why the analytic integral forms of Lame functions can not be obtained, is that the coefficients in a power series expansions

do not have two term recursion relations. We have a relation between three different coefficients. By using the three term recurrence formula[1], we are able to obtain an analytic integral solution of any linear ordinary differential equation in which has three term recursion relations.

7 Series “Special functions and three term recurrence formula (3TRF)”

This paper is 7th out of 10.

1. “Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system” [14] - In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the Grand Confluent Hypergeometric Function. Our new solution produces previously unknown extra hidden quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

2. “Generalization of the three-term recurrence formula and its applications” [15] - Generalize three term recurrence formula in linear differential equation. Obtain the exact solution of the three term recurrence for polynomials and infinite series.

3. “The analytic solution for the power series expansion of Heun function” [16] - Apply three term recurrence formula to the power series expansion in closed forms of Heun function (infinite series and polynomials) including all higher terms of A_n s.

4. “Asymptotic behavior of Heun function and its integral formalism”, [17] - Apply three term recurrence formula, derive the integral formalism, and analyze the asymptotic behavior of Heun function (including all higher terms of A_n s).

5. “The power series expansion of Mathieu function and its integral formalism”, [18] - Apply three term recurrence formula, analyze the power series expansion of Mathieu function and its integral forms.

6. “Lamé equation in the algebraic form” [19] - Applying three term recurrence formula, analyze the power series expansion of Lamé function in the algebraic form and its integral forms.

7. “Power series and integral forms of Lamé equation in the Weierstrass's form” [20] - Applying three term recurrence formula, derive the power series expansion of Lamé function in the Weierstrass's form and its integral forms.

8. “The generating functions of Lamé equation in the Weierstrass's form” [21] - Derive the generating functions of Lamé function in the Weierstrass's form (including all higher terms of A_n 's). Apply integral forms of Lamé functions in the Weierstrass's form.

9. “Analytic solution for grand confluent hypergeometric function” [22] - Apply three term recurrence formula, and formulate the exact analytic solution of grand

confluent hypergeometric function (including all higher terms of A_n 's). Replacing μ and $\varepsilon\omega$ by 1 and $-q$, transforms the grand confluent hypergeometric function into Biconfluent Heun function.

10. "The integral formalism and the generating function of grand confluent hypergeometric function" [23] - Apply three term recurrence formula, and construct an integral formalism and a generating function of grand confluent hypergeometric function (including all higher terms of A_n 's).

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